4.1 3D GEOMETRIC TRANSFORMATIONS

Methods for geometric transformations and object modeling in three dimensions are extended from two-dimensional methods by including considerations for the ***z*** coordinate. We now translate an object by specifying a three-dimensional translation vector, which determines how much the object is to be moved in each of the three coordinate directions. Similarly, we scale an object with three coordinate scaling factors. The extension for three-dimensional rotation is less straightforward. When we discussed two-dimensional rotations in the *xy* plane, we needed to consider only rotations about axes that were perpendicular to the *xy* plane. In three-dimensional space, we can now select any spatial orientation for the rotation axis. Most graphics packages handle three-dimensional rotation as a composite of three rotations, one for each of the three Cartesian axes. Alternatively, a user can easily set up a general rotation matrix, given the orientation of the axis and the required rotation angle. As in the two-dimensional case, we express geometric transformations in matrix form. Any sequence of transformations is then represented as, a single matrix, formed by concatenating the matrices for the individual transformations in the sequence.

Transformation Matrix

$$\begin{bmatrix} A & D & G & J \\ B & E & H & K \\ C & F & I & L \\ 0 & 0 & 0 & S \end{bmatrix} \Rightarrow \left[\begin{array}{c|c} 3\times3 & 3\times1 \\ \hline 1\times3 & 1\times1 \end{array}\right]$$

3×3 : Scaling, Reflection, Shearing, Rotation

3×1 : Translation

1×1 : Uniform global Scaling

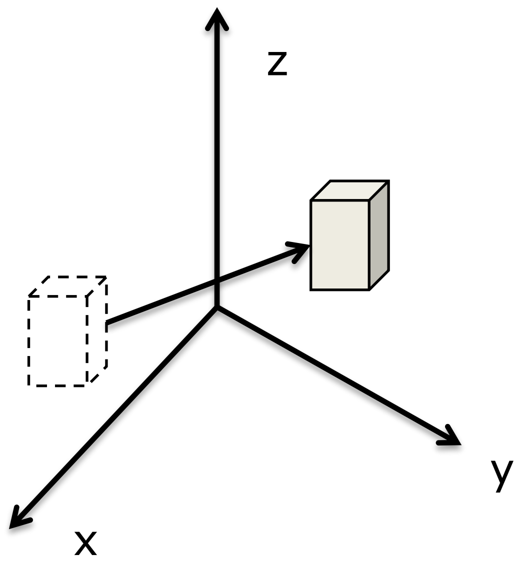
1×3 : Homogeneous representation

4.1.1 TRANSLATION

In a three-dimensional homogeneous coordinate representation, a point is translated (Fig. **4.1)** from position P = ***(x,* y, *z)*** to position P' = ***(x'*, y',** z') with the matrix operation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$x' = x + t_x, \quad y' = y + t_y, \quad z' = z + t_z$$



(Fig: 4.1)

An object is translated in three dimensions by transforming each of the defining points of the object. For an object represented as a set of polygon surfaces, we translate each vertex of each surface and redraw the polygon facets in the new position.

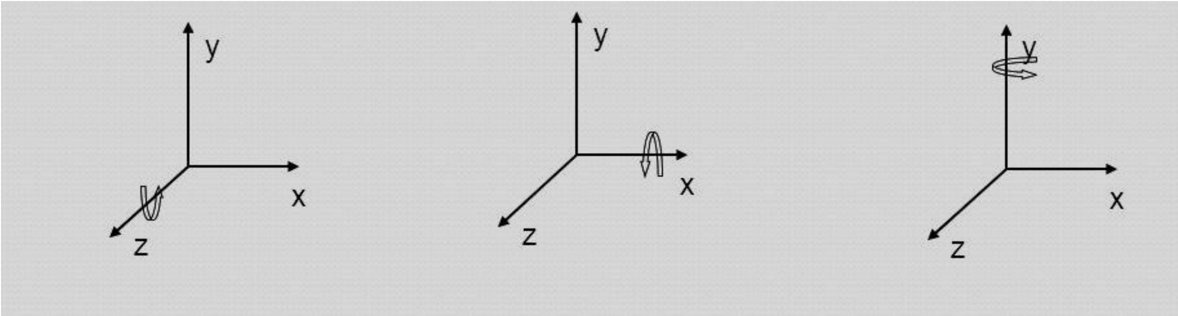
We obtain the inverse of the translation matrix in the given equation by negating the translation distances t_x, t_y, and t_z. This produces a translation in the opposite direction, and the product of a translation matrix and its inverse produces the identity matrix.

4.1.2 ROTATION

To generate a rotation transformation for an object, we must designate an axis of rotation (about which the object is to be rotated) and the amount of angular rotation. Unlike two-dimensional applications, where all transformations are carried out in the **xy** plane, a three-dimensional rotation can be specified around any line in space. The easiest rotation axes to handle are those that are parallel to the coordinate axes. Also, we can use combinations of coordinate axis rotations (along with appropriate translations) to specify any general rotation. By convention, positive rotation angles produce counterclockwise rotations about a coordinate axis, if we are looking along the positive half of the axis toward the coordinate origin this agrees with our earlier discussion of rotation in two dimensions, where positive rotations in the **xy** plane are counterclockwise about axes parallel to the z axis.

Coordinate-Axes Rotations

- **X-axis rotation**
- **Y-axis rotation**
- **Z-axis rotation**



- **X-Axis Rotation**

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- **Z-Axis Rotation**

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- **Y-Axis Rotation**

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

An inverse rotation matrix is formed by replacing the rotation angle**.** Negative values for rotation angles generate rotations in a clockwise direction, so the identity matrix is produced when any rotation matrix is multiplied by is inverse. Since only the sine function is affected by the change in sign of the rotation angle, the inverse matrix can also be obtained by interchanging rows and columns. That is, we can calculate the inverse of any rotation matrix R by evaluating its transpose ($R^{-1} = R^T$). This method for obtaining an inverse matrix holds also for any composite rotation matrix.

4.1.2.1 General Three-Dimensional Rotations

A rotation matrix for any axis that does not coincide with a coordinate axis can be set up as a composite transformation involving combinations of translations and the coordinate-axes rotations. We obtain the required composite matrix by first setting up the transformation sequence that moves the selected rotation axis onto one of the coordinate axes. Then we set up the rotation matrix about that coordinate axis for the specified rotation angle. The last step is to obtain the inverse transformation sequence that returns the rotation axis to its original position.

In the special case where an object is to be rotated about an axis that is parallel to one of the coordinate axes, we can attain the desired rotation with the following transformation sequence.

1) Translate the object so that the rotation axis coincides with the parallel coordinate axis.

2) Perform the specified rotation about that axis.

3) Translate the object so that the rotation axis is moved back to its original position.

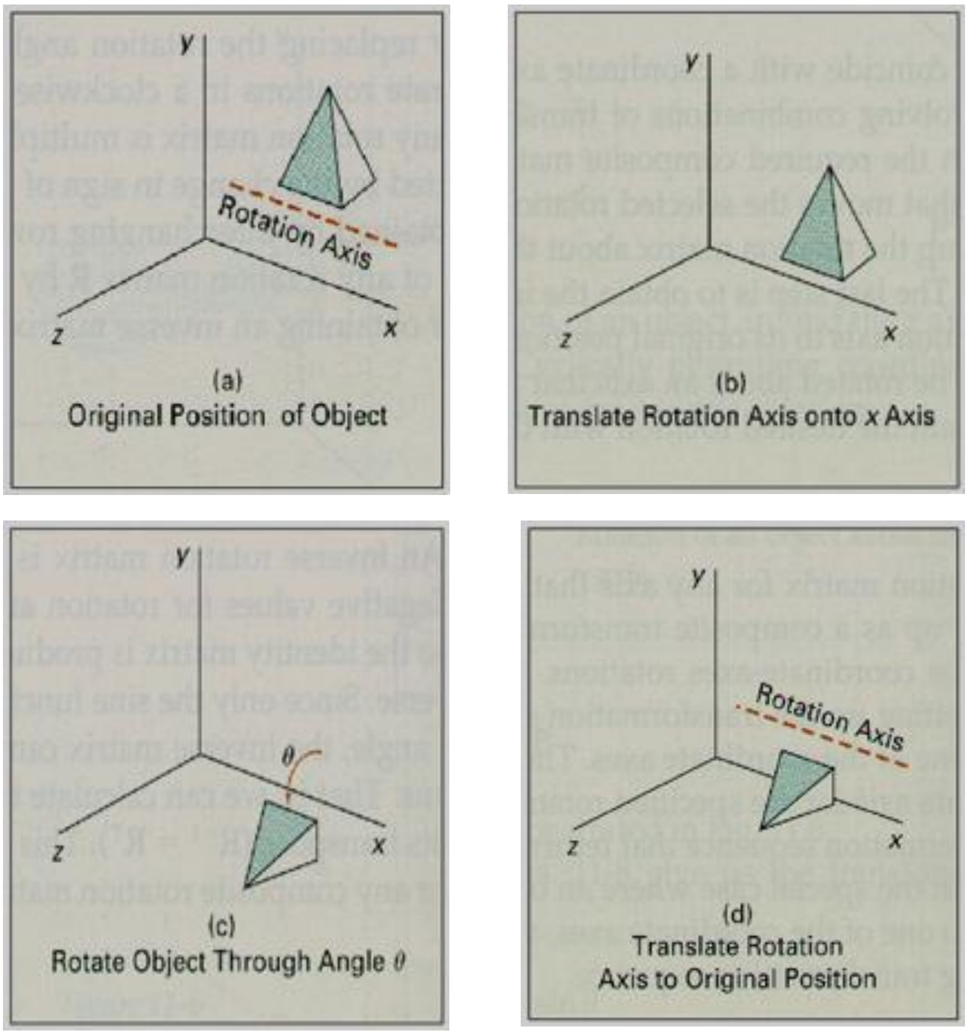
The steps in this sequence are illustrated in **Fig. 4-2**. Any coordinate position **P** on the object in this figure is transformed with the sequence shown as

$$\mathbf{P' = t^{-1} . R_z (\theta) . T . P}$$

Where the composite matrix for the transformation is

$$\mathbf{R (\theta) = T^{-1} . R_z (\theta) . T}$$

This is of the same as the two-dimensional transformation sequence for rotation about an arbitrary pivot point.



(Fig : 4.2)

When an object is to be rotated about an axis that is not parallel to one of the coordinate axes, we need to perform some additional transformations. In this case, we also need rotations to align the axis with a selected coordinate axis and to bring the axis back to its original orientation

Given the specifications for the rotation axis and the rotation angle, we can accomplish the required rotation in five steps

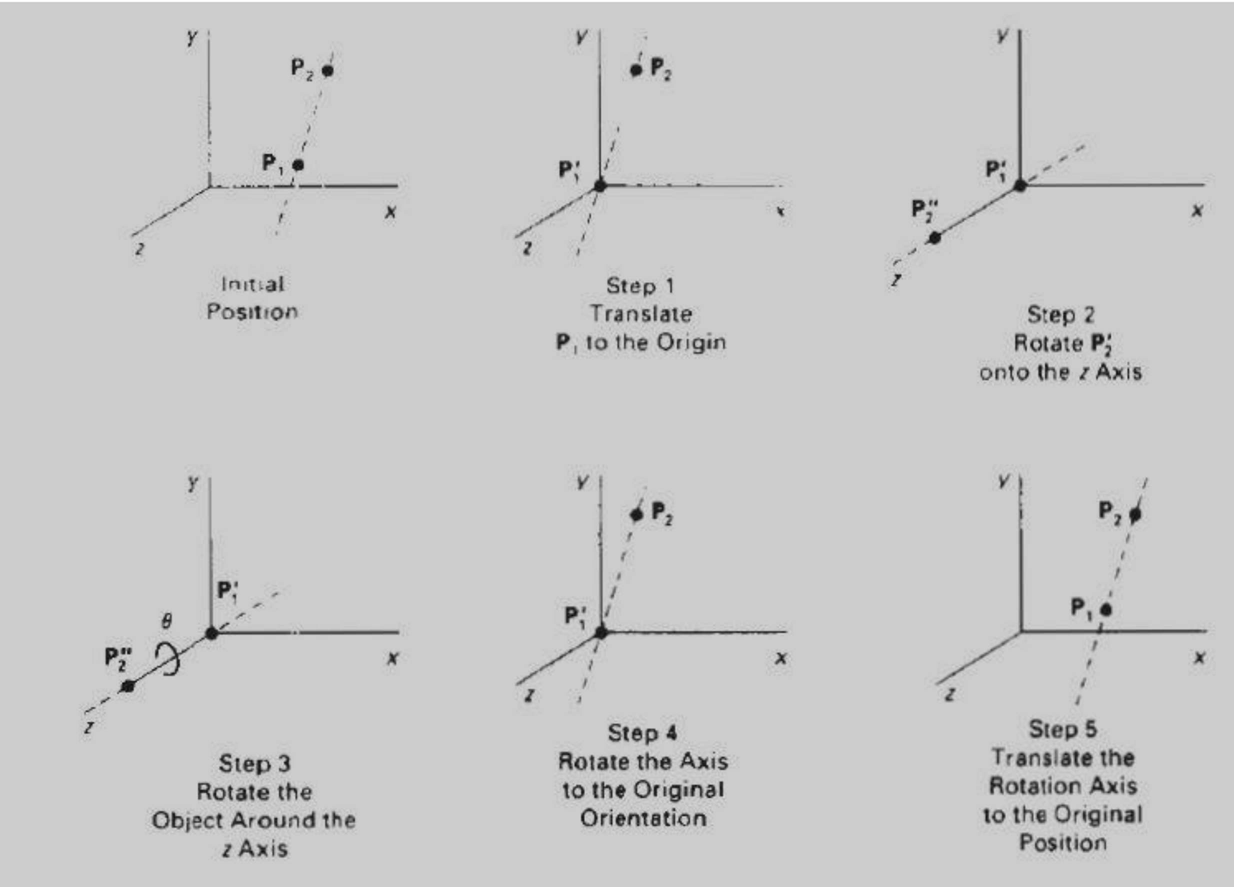
1) Translate the object so that the rotation axis pass= through the coordinate origin.

2) Rotate the object so that the axis of rotation coincides with one of the coordinate axes.

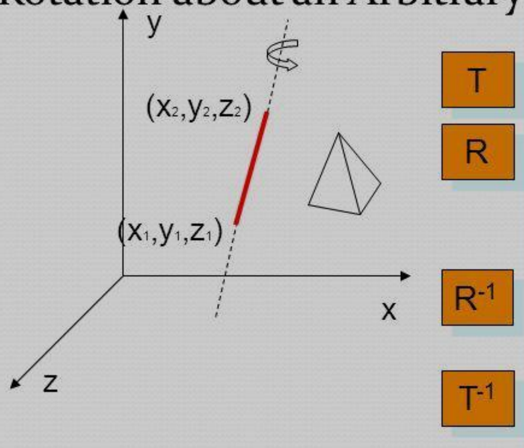
3) Perform the specified rotation about that coordinate axis.

4) Apply inverse rotations to bring the rotation axis back to its original orientation.

5) Apply the inverse translation to bring the rotation axis back to its original position.



- Rotation about an Arbitrary Axis

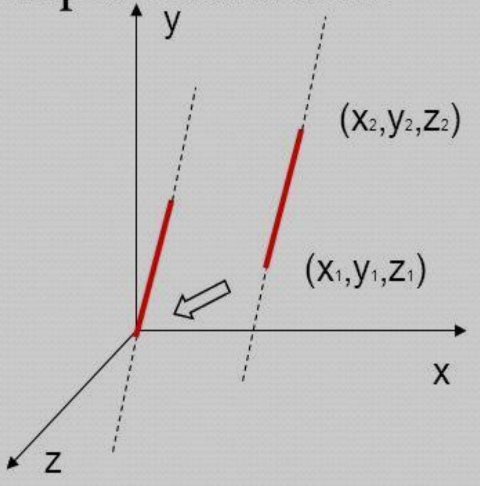


Basic Idea

1. Translate (x1, y1, z1) to the origin
2. Rotate (x'2, y'2, z'2) on to the z axis
3. Rotate the object around the z-axis
4. Rotate the axis to the original orientation
5. Translate the rotation axis to the original position

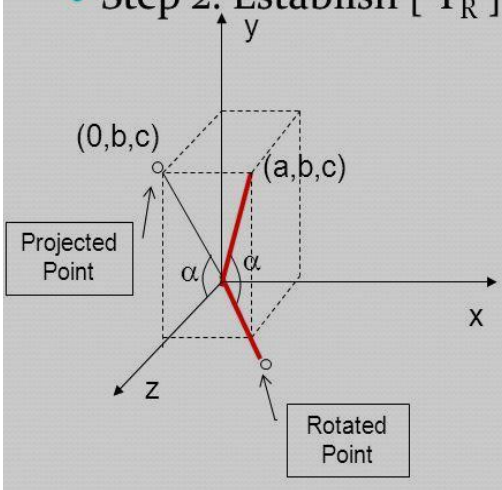
$$\mathbf{R}(\theta)=\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

- Step 1. Translation



$$\mathbf{T}=\begin{bmatrix}1 & 0 & 0 & -x_1\\ 0 & 1 & 0 & -y_1\\ 0 & 0 & 1 & -z_1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

- Step 2. Establish $[\,T_R\,]^{\alpha}_{x}$ *x* axis

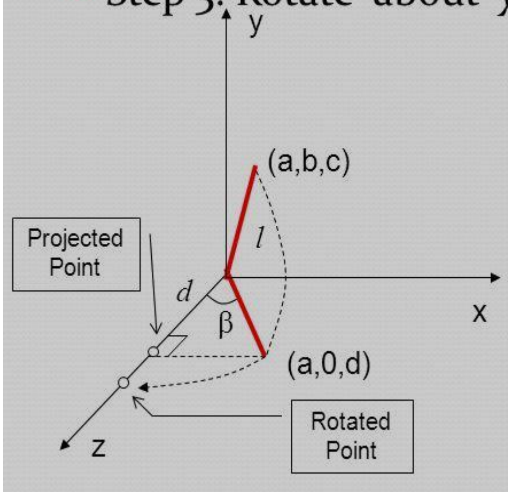


$$\sin\alpha=\frac{b}{\sqrt{b^2+c^2}}=\frac{b}{d}$$

$$\cos\alpha=\frac{c}{\sqrt{b^2+c^2}}=\frac{c}{d}$$

$$\mathbf{R}_x(\alpha)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha & 0\\ 0 & \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & c/d & -b/d & 0\\ 0 & b/d & c/d & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

- Step 3. Rotate about *y* axis by ϕ



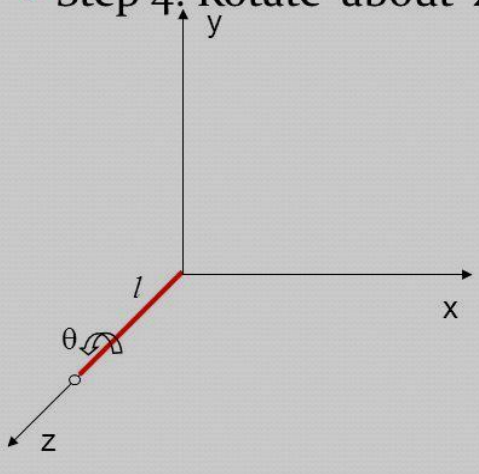
$$\sin\beta = \frac{a}{l}, \qquad \cos\beta = \frac{d}{l}$$

$$l^2 = a^2 + b^2 + c^2 = a^2 + d^2$$

$$d = \sqrt{b^2 + c^2}$$

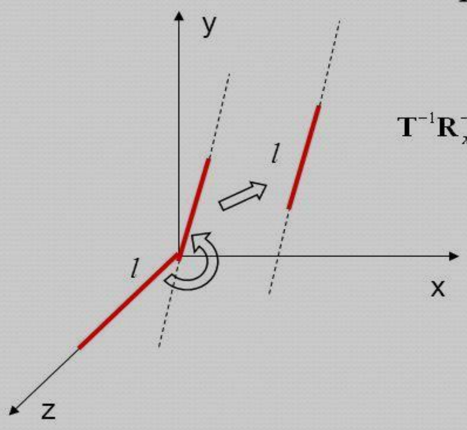
$$\mathbf{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d/l & 0 & -a/l & 0 \\ 0 & 1 & 0 & 0 \\ a/l & 0 & d/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 4. Rotate about *z* axis by the desired angle θ



$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 5. Apply the reverse transformation to place the axis back in its initial position



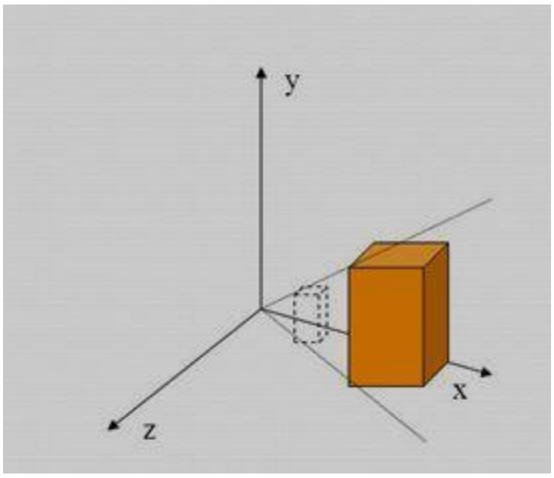
$$\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta) = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}(\theta) = \mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

4.1.3 SCALING

The matrix expression tor the scaling transformation of a position P = **(x,** y, *z)* relative to the coordinate origin can be written as

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



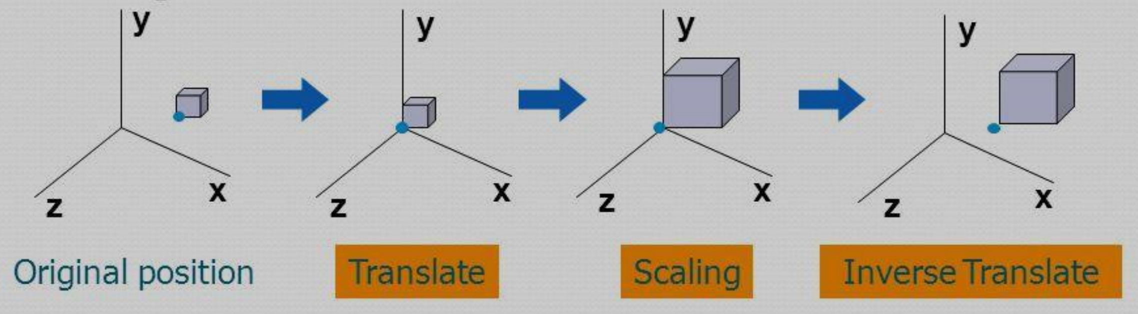
Where scaling parameters s_x, $\mathbf{s_y}$, and $\mathbf{s_z}$ are assigned any positive values. Explicit expressions for the coordinate transformations for scaling relative to the origin are

$$x' = x \cdot s_x, \quad y' = y \cdot s_y, \quad z' = z \cdot s_z$$

Scaling with respect to a selected fixed position **(x,** y**,** ***z)*** can be represented with the following transformation sequence:

1) **Translate the fixed point to the origin.**
2) **Scale the object relative to the coordinate origin.**
3) **Translate the fixed point back to its original position.**

- Scaling with a Selected Fixed Position



$$T(x_f,y_f,z_f)\cdot S(s_x,s_y,s_z)\cdot T(-x_f,-y_f,-z_f)=\begin{bmatrix}x'\\y'\\z'\\1\end{bmatrix}=\begin{bmatrix}1&0&0&x_f\\0&1&0&y_f\\0&0&1&z_f\\0&0&0&1\end{bmatrix}\begin{bmatrix}s_x&0&0&0\\0&s_y&0&0\\0&0&s_z&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&-x_f\\0&1&0&-y_f\\0&0&1&-z_f\\0&0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}$$

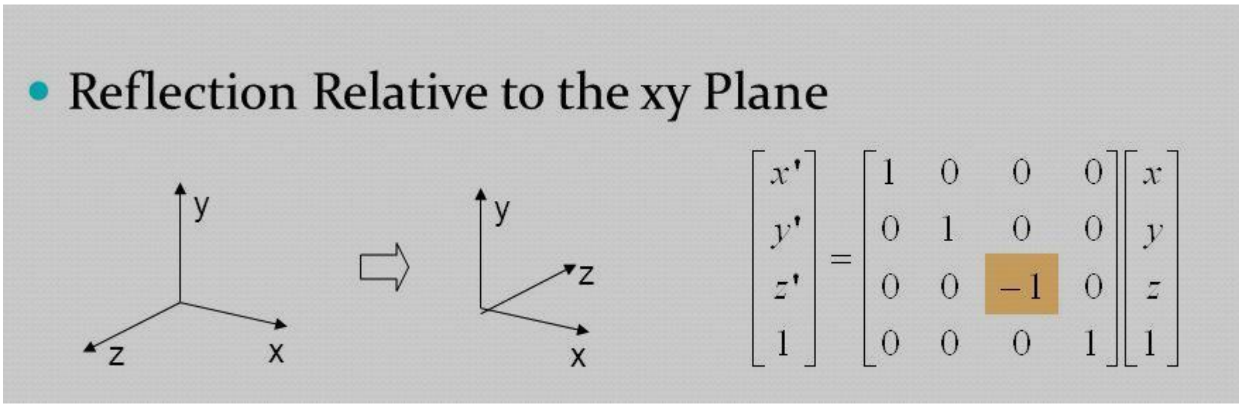
4.2 OTHER TRANSFORMATIONS

In addition to translation, rotation, and scaling, there are various additional transformations that are often useful in three-dimensional graphics applications. Two of these are reflection and shear.

4.2.1 REFLECTIONS

A three-dimensional reflection can be performed relative to a selected ***reflection axis*** or with respect to a selected ***reflection plane.*** In general, three-dimensional reflection matrices are set up similarly to those for two dimensions. Reflections relative to a given axis are equivalent to 180^o rotations about that axis. Reflections with respect to a plane are equivalent to 180^o rotations in four-dimensional space. When the reflection plane is a coordinate plane (either **xy,** ***xz, or*** yz), we can think of the transformation as a conversion between Left-handed and right-handed systems.

An example of a reflection that converts coordinate specifications from a right-handed system to a left-handed system (or vice versa) is shown in **Fig. 4-3**. This transformation changes the sign of the z coordinates, Leaving the ***x*** and y-coordinate values unchanged. The matrix representation for this reflection of points relative to the **xy** plane is given below



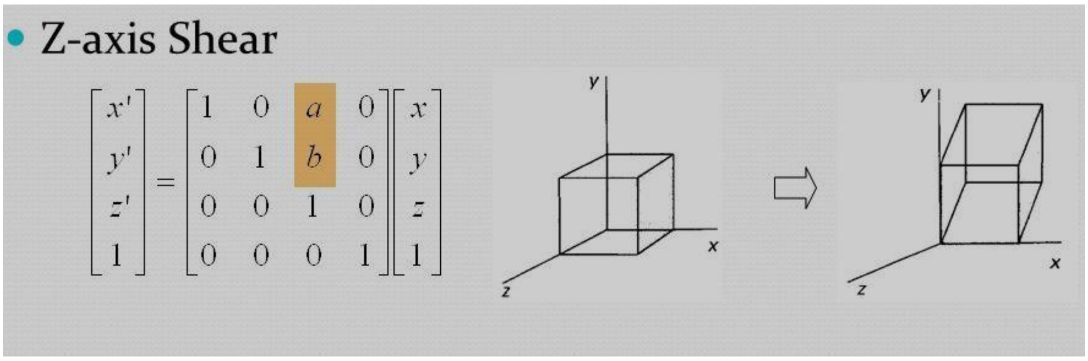
(Fig: 4.3)

Transformation matrices for inverting **x** and y values are defined similarly, as reflections relative to the yz plane and ***xz*** plane, respectively Reflections about other planes can be obtained as a combination of rotations and coordinate-plane reflections.

4.2.2 SHEARS

Shearing transformations can he used to modify object shapes. They are also useful in three-dimensional viewing for obtaining general projection transformations. In two dimensions, we discussed transformation relative to the **x** or y axes to produce distortions in the shapes of objects. In three dimensions, we can also generate shears relative to the *z* axis.

As an example of three-dimensional shearing the following transformation produces a z-axis shear:



(Fig: 4.4)

Parameters a and *b* can be assigned any real values. The effect of this transformation matrix is to alter **x-** and y-coordinate values by an amount that is proportional to the **z** value, while leaving the ***z*** coordinate unchanged. Boundaries of planes that are perpendicular to the ***z*** axis are thus shifted by an amount proportional to ***z.*** An example of the effect of this shearing matrix on a unit cube is shown in **Fig. 4-4**, for shearing values a = *b* = 1. Shearing matrices for the x axis and y axis are defined similarly.