3.1 THREE DIMENSIONAL CONCEPTS

We can rotate an object about an axis with any spatial orientation in three-dimensional space. Two-dimensional rotations, on the other hand, are always around an axis that is perpendicular to the ***xy*** plane. Viewing transformations in three dimensions are much more cornplicated because we have many more parameters to select when specifying how a three-dimensional scene is to be mapped to a display device. The scene description must be processed through viewing-coordinate transformations and projection routines that transform three-dimensional viewing coordinates onto two-dimensional device coordinates. Visible parts of a scene, for a selected view, must be identified; and surface-rendering algorithms must he applied if a realistic rendering of the scene is required.

3.2 THREE DIMENSIONAL OBJECT REPRESENTATIONS

Graphics scenes contain many different kinds of objects. Trees, flowers, glass, rock, water etc.. There is not any single method that we can use to describe objects that will include all characteristics of these different materials.

Polygon and quadric surfaces provide precise descriptions for simple Euclidean objects such as polyhedrons and ellipsoids.

Spline surfaces and construction techniques are useful for designing aircraft wings, gears and other engineering structure with curved surfaces.

Procedural methods such as fractal constructions and particle systems allow us to give accurate representations for clouds, clumps of grass and other natural objects.

Physically based modeling methods using systems of interacting forces can be used to describe the non-rigid behavior of a piece of cloth or a glob of jello.

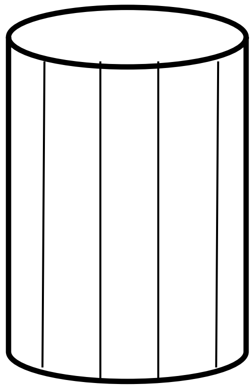
Octree encodings are used to represent internal features of objects; such as those obtained from medical CT images.

Isosurface displays, volume renderings and other visualization techniques are applied to 3 dimensional discrete data sets to obtain visual representations of the data.

3.3 POLYGON SURFACES

Polygon surfaces provide precise descriptions for simple Euclidean objects such as polyhedrons and ellipsoids. A three dimensional graphics object can be represented by a set of surface polygons. Many graphic systems store a 3 dimensional object as a set of surface polygons. This simplifies and speeds up the surface rendering and display of objects. In this representation, the surfaces are described with linear equations. The polygonal representation of a polyhedron precisely defines the surface features of an object.

In **Figure 3.1**, the surface of a cylinder is represented as a polygon mesh. Such representations are common in design and solid- modeling applications, since the wireframe outline can be displayed quickly to give a general indication of the surface structure.



(Figure 3.1, Wireframe representation of a cylinder with back (hidden) lines removed)

3.4 POLYGON TABLES

We know that a polygon surface is defined by a set of vertices. As information for each polygon is input, the data are placed in to tables that are used in later processing, display and manipulation of objects in the scene.

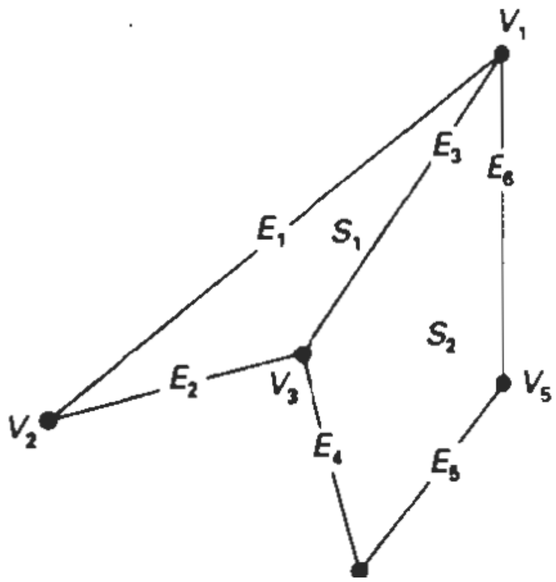
Polygon data tables can be organized in to 2 groups: geometric tables and attribute tables. Geometric data tables contain vertex coordinates and parameters to identify the spatial orientation of the polygon surfaces. Attribute information for an object includes parameters specifying the degree of transparency of the object and its surface reflexivity and texture characteristics.

A suitable organization for storing geometric data is to create 3 lists, a vertex table, an edge table and a polygon table. Coordinate values for each vertex is stored in the vertex table. The edge table contains pointers back to the vertex table to identify the vertices for each polygon edge. The polygon table contains pointers back to the edge table to identify the edges for each polygon.

An alternative arrangement is to modify the edge table to include forward pointers in to the polygon table so that common edges between polygons could be identified more rapidly.

Additional geometric formation that is usually stored in the data tables includes the slope for each edge and the coordinate extents for each polygon. As vertices are input, we can calculate edge slopes, and we can scan the coordinate values to identify the minimum and maximum ***x***, y, and ***z*** values for individual polygons. Edge slopes and bounding-box information for the polygons are needed in subsequent processing, for example, surface rendering. Coordinate extents are also used in some visible-surface determination algorithms

The more information included in the data tables, the easier it is to check for errors. Therefore, error checking is easier when three data tables (vertex, edge, and polygon) are used, since this scheme provides the most information. Some of the tests that could be performed by a graphics package are **(1)** that every vertex is listed as an endpoint for at least two edges, **(2)** that every edge is part of at least one polygon, **(3)** that every polygon is closed, **(4)** that each polygon has at least one shared edge, and **(5)** that if the edge table contains pointers to polygons, every edge referenced by a polygon pointer has a reciprocal pointer back to the polygon.



VERTEX TABLE
V1:x1,y1,z1
V2:x2,y2,z2
V3:x3,y3,z3
V4:x4,y4,z4
V5:x5,y5,z5

EDGE TABLE
E1:V1,V2
E2:V2,V3
E3:V3,V1
E4:V3,V4
E5:V4,V5
E6:V5,V1

POLYGON-SURFACE TABLE
S1:E1,E2,E3
S2:E3,E4,E5,E6

3.5 PLANE EQUATIONS

When working with polygons or polygon meshes, we need to know the equation of the plane in which the polygon lies. We can use the coordinates of 3 vertices to find the plane. The plane equation is

$$Ax + By + Cz + D = 0$$

The coefficients A, B and C define the normal to the plane. [A B C]. We can obtain the coefficients A, B, C and D by solving a set of 3 plane equations using the coordinate values for 3 non collinear points in the plane. Suppose we have 3 vertices on the polygon (x1, y1, z1), (x2, y2, z2) and (x3, y3, z3).

$$Ax + By + Cz + D = 0$$

$$(A/D)\ x1 + (B/D)\ y1 + (C/D)\ z1 = -1$$

$$(A/D)\ x2 + (B/D)\ y2 + (C/D)\ z2 = -1$$

$$(A/D)\ x3 + (B/D)\ y3 + (C/D)\ z3 = -1$$

the solution for this set of equations can be obtained using Cramer's rule as,

$$A = \begin{matrix} 1 & y1 & z1 \\ 1 & y2 & z2 \\ 1 & y3 & z3 \end{matrix}$$

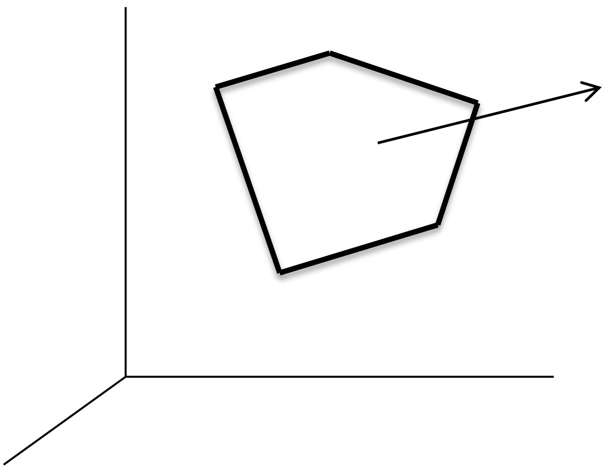
$$B = \begin{matrix} x1 & 1 & z1 \\ x2 & 1 & z2 \\ x3 & 1 & z3 \end{matrix}$$

$$C = \begin{matrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \end{matrix}$$

$$D = - \begin{matrix} x1 & y1 & z1 \\ x2 & y2 & z2 \\ x3 & y3 & z3 \end{matrix}$$

We can write the calculations for the plane coefficients in the form

$$A = y1\,(z2 - z3) + y2\,(z3 - z1) + y3\,(z1 - z2)$$
$$B = z1\,(x2 - x3) + z2\,(x3 - x1) + z3\,(x1 - x2)$$
$$C = x1\,(y2 - y3) + x2\,(y3 - y1) + x3\,(y1 - y2)$$
$$D = - x1\,(y2\,z3 - y3\,z2) - x2\,(y3\,z1 - y1\,z3) - x3\,(y1\,z2 - y2\,z1)$$



(Figure 3.2, The vector N, Normal to the surface of a plane described by the equation Ax + By + Cz + D = 0 ,has Cartesian components(A,B,C))

If there are more than 3 vertices, the polygon may be non-planar. We can check whether a polygon is nonplanar by calculating the perpendicular distance from the plane to each vertex. The distance d for the vertex at (x, y, z) is

$$d = \frac{Ax + By + Cz + D}{(A^2 + B^2 + C^2)}$$

The distance is either positive or negative, depending on which side of the plane the point is located. If the vertex is on the plane, then d = 0.

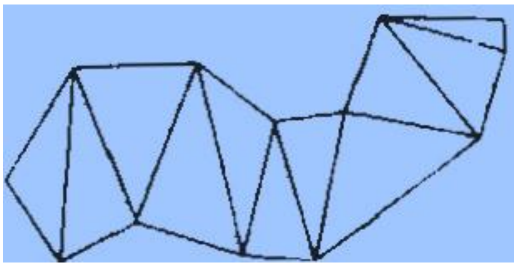
We can identify the point as either inside or outside the plane surface according to the sign of Ax + By + Cz + D.

If Ax + By + Cz + D < 0, the point (x, y, z) is inside the surface.

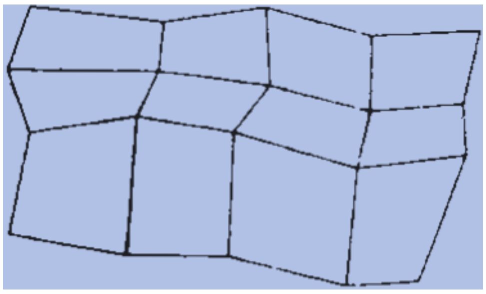
If Ax + By + Cz + D > 0, the point (x, y, z) is outside the surface.

3.6 POLYGON MESHES

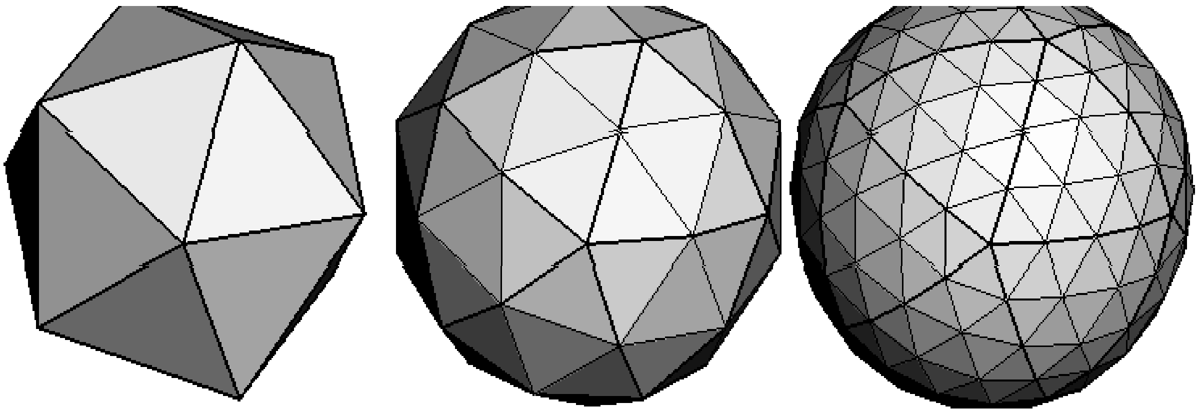
One type of polygon mesh is the **triangle strip**. This function produces ***n* - **2 connected triangles, **as** shown in Figure: 3.3, given the coordinates for ***n*** vertices. Another similar function is the **quadrilateral mesh,** which generates a mesh of ***(n* -** I) by **(m -** 1) quadrilaterals, given the coordinates for an **n** by m array of vertices. Figure: 3.4 shows 20 vertices forming a mesh of 12 quadrilaterals



(Figure: 3.3 A triangle strip formed with a 11 triangles connecting 13 vertices)



(Figure: 3.4 shows 20 vertices forming mesh of 12 quadrilaterals)



(Figure: 3.5, Examples for triangle mesh with number of vertices)

High quality graphics systems typically model objects with polygon meshes and set up a database of geometric and attribute information to facilitate processing of the polygon facets. Fast hardware-implemented polygon renderers are incorporated into such systems with the capability for displaying hundreds of thousands to one million br more shaded polygons per second (usually triangles), including the application of surface texture and special lighting effects.

(Figure: 3.6, Example for triangle mesh for an image)

3.7 QUADRIC SURFACES

These are frequently used class of objects. Quadric surfaces are described with second-degree equations. (Quadratics)

The examples of quadric surfaces are spheres, ellipsoids, tori, paraboloids and hyperboloids.

Spheres and ellipsoids are common elements of graphic scenes.

Sphere

A spherical surface with radius r centered on the coordinate origin is defined as the set of points (x, y, z) that satisfy the equation

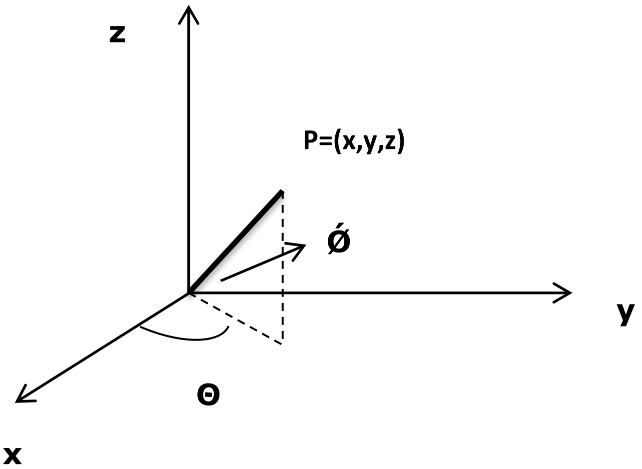
$$x^2 + y^2 + z^2 = r^2$$

in parametric form, the equation of a sphere is

$x = r. \cos \acute{\varnothing}. \cos\theta \qquad -\prod/2 <= \acute{\varnothing} <= \prod/2$

$y = r. \cos \acute{\varnothing}. \sin \theta \qquad -\prod <= \acute{\varnothing} <= \prod$

$z = r. \sin \acute{\varnothing}$

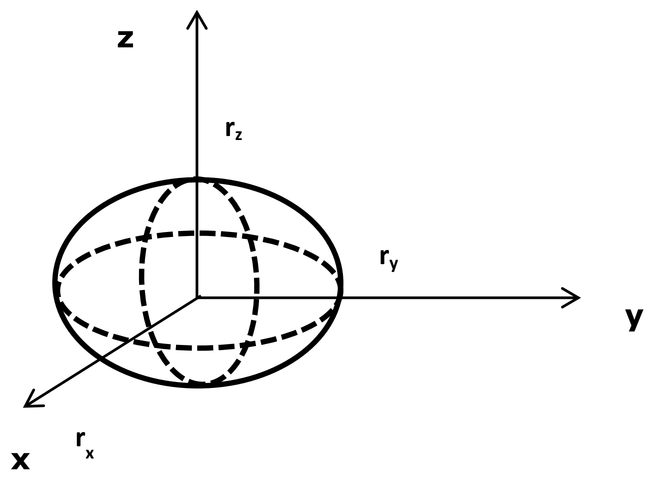


(Figure: 3.7, Parametric coordinate position(r,θ,Ǿ) on the surface of a sphere with radius r)

Ellipsoid

An ellipsoidal surface is described as an extension for a spherical surface, where the radii in 3 mutually perpendicular directions can have different values. The points over the surface of an ellipsoid centered at the origin is

$$(x/r_x)^2 + (y/r_y)^2 + (z/r_z)^2 = 1$$



(Figure: 3.8, An ellipsoid with radii r_x, r_y, and r_z centered on the coordinate origin)

The parametric representation for the ellipsoid in terms of the latitude angle Ǿ and the longitude angle θ in Figure: 3.8 is

$x=r_x \cos\acute{Ø}\cos\theta$ $\quad -\prod/2 <= \acute{Ø} <= \prod/2$
$y=r_y \cos\acute{Ø}\sin\theta$ $\quad -\prod <= \acute{Ø} <= \prod$
$z = r_z . \sin \acute{Ø}$

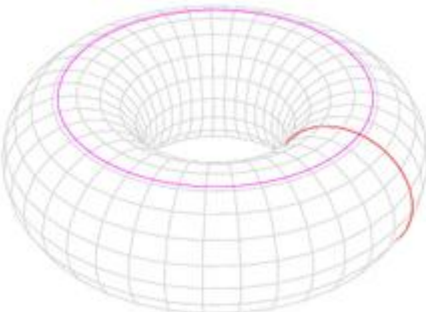
Torus

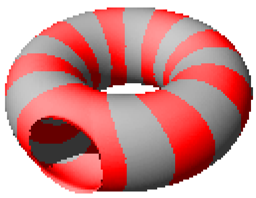
The torus is a doughnut shaped object. It can be generated by rotating a circle or other conic about a specified axis. The Cartesian representation for points over the surface of a torus can be written as

$$\left(r - \sqrt{(x/rx)^2 + (y/ry)^2}\right)^2 + (z/rz)^2 = 1$$

We can describe the parametric representation of a torus are similar to those for an ellipse, except that angle Ǿ extends over 360^o. The parametric representation for the etorus in terms of the latitude angle Ǿ and the longitude angle θ

$x=r_x(r+ \cos\acute{Ø})\cos\theta,$ $\quad -\prod<= \acute{Ø} <= \prod$
$y=r_y (r+\cos\acute{Ø})\sin\theta,$ $\quad -\prod <= \acute{Ø} <= \prod$
$z = r_z . \sin \acute{Ø}$





3.8 SUPER QUADRATICS

These objects are a generalization of the quadric representations. Super quadratics is formed by incorporating additional parameters in to the quadric equations. It is for providing increased flexibility for adjusting object shapes.

Super ellipse

The Cartesian representation for a super ellipse is obtained from the equation of an ellipse by allowing the exponent on the x and y terms to be variable.

The equation of a super ellipse is

$$\left(\frac{x}{r_x}\right)^{2/s} + \left(\frac{y}{r_y}\right)^{2/s} = 1$$

The parameter s can be assigned any real value. When s = 1, we get an ordinary ellipse. Corresponding parametric equations for the superellipse can be expressed as

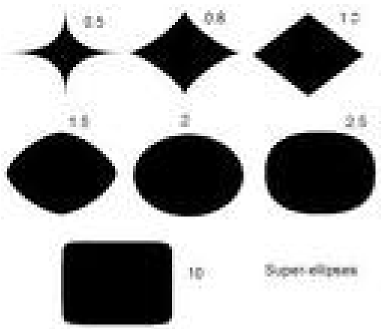
$x = r_x \cos^s \theta$, $-\prod <= Ǿ <= \prod$

$y = r_y \sin^s \theta$,

Figure 3.9 illustrates super circle shapes that can be generated using various values for parameter s.

.5 1.0 1.5 2.0 2.3 3.0

(Figure: 3.9 Super ellipses plotted with different values for parameter s and with $r_x = r_y$)



Super ellipsoid

The Cartesian representation for a super ellipsoid is obtained from the equation of an ellipsoid.

$$\left[\left(\frac{x}{r_x}\right)^{2/s2} + \left(\frac{y}{r_y}\right)^{2/s2}\right]^{\frac{s2}{s1}} + \left(\frac{z}{r_z}\right)^{2/s1} = 1$$

For s1 = s2 = 1, we will get an ellipsoid.

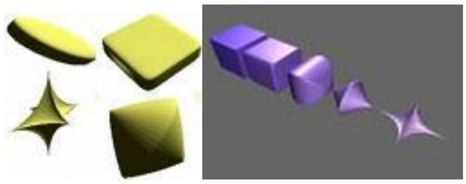
The parametric equations for super ellipsoid are

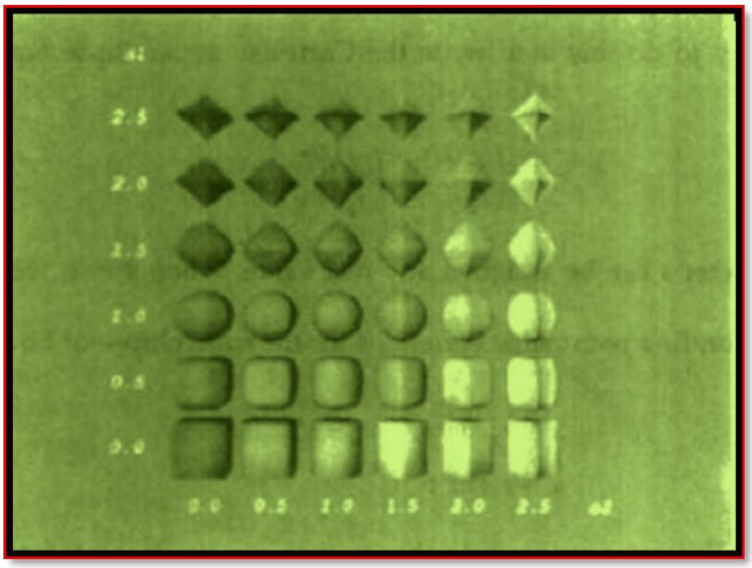
$x = r_x \cos^{s1} Ǿ \cos^{s2} \theta$, $-\prod/2 <= Ǿ <= \prod/2$

$y = r_y \cos^{s1} Ǿ \sin^{s2} \theta$, $-\prod <= \theta <= \prod$

$z = r_z \sin^{s1} Ǿ$

These shapes can be combined to create more complex structures such as furniture, threaded bolts and other hardware.





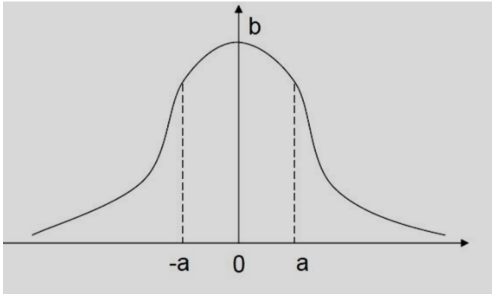
(Figure: 3.10 Super ellipsoids plotted with different values for parameters **s1** and ***s2*** and with $\mathbf{r_x} = \mathbf{r_y} = \mathbf{r_z}$**).**

3.9 BLOBBY OBJECTS

Some objects do not maintain a fixed shape, but change their surface characteristics in certain motions or when in proximity to other objects. Examples in this class of objects include molecular structures, water droplets and other liquid effects, melting objects, and muscle shapes in the human body. These objects can be described as exhibiting "blobbiness" and are often simply referred to as blobby objects, since their shapes show a certain degree of fluidity.

A molecular shape, for example, can be described as spherical in isolation, but this shape changes when the molecule approaches another molecule. This distortion of the shape of the electron density cloud is due to the "bonding" that occurs between the two molecules. **Figure: 3.11** illustrate the stretching, snap ping, and contracting effects on molar shapes when two molecules move apart. These characteristics cannot be adequately described simply with spherical or elliptical shapes. Similarly, **Figure: 3.12** shows muscle shapes in a human arm, which exhibit similar characteristics.

Several models have been developed for representing blobby objects as distribution functions over a region of space. One way to do this is to model objects as combinations of Gaussian density functions, or "bumps" (**Figure: 3.13**).



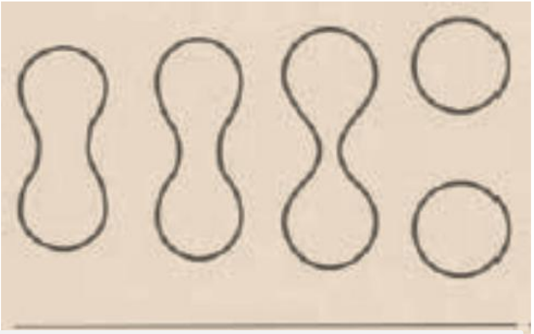


Figure: 3.11

Molecular bonding. As two molecules move away from each other, the surface shapes stretch, snap, and finally contract into spheres.

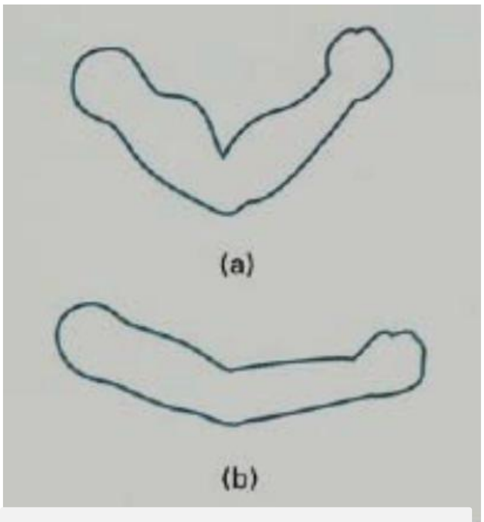


Figure: 3.12

Blobby muscle shapes in a human arm.

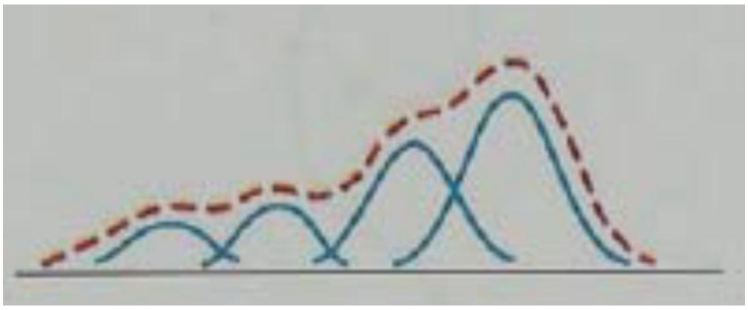
A surface function is then defined as

$$f(x,y,z) = \sum_k f(r_k) - T$$

$$r_k = \sqrt{(x-x_k)^2 + (y-y_k)^2 + (z-z_k)^2}$$

Where $f(r_k) = b_k e^{-a_k r_k^2}$ or $f(r_k) = b_k e^{-a_k r_k}$

Parameter *T* is some specified threshold, and parameters *a* and *b* are used to adjust the amount of blobbiness of the individual objects. (Negative *b*–dents)



(Figure: 3.13, A composite blobby object formed with four Gaussian bumps)

Other methods for generating blobby objects use density functions that fall off to 0 in a finite interval, rather than exponentially.

- The "metaball" model:

$$f(r) = \begin{cases} b(1-3r^2/d^2), & (0 < r \le d/3) \\ 1.5b(1-r/d)^2, & (d/3 < r \le d) \\ 0, & (r > d) \end{cases}$$

- The "soft object" model:

$$f(r) = \begin{cases} 1-\dfrac{22r^2}{9d^2}+\dfrac{17r^4}{9d^4}-\dfrac{4r^6}{9d^6}, & (0 < r \le d) \\ 0, & (r > d) \end{cases}$$

3.10 SPLINE REPRESENTATIONS

In computer graphics, the term spline curve now refer to any composite curve formed with polynomial sections satisfying specified continuity conditions at the boundary of the pieces.

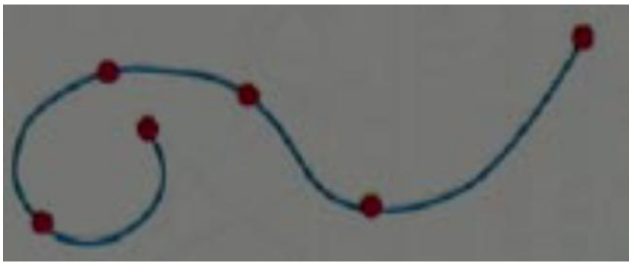
A spline surface can be described with two sets of orthogonal spline curves.

There are several different kinds of spline specifications – each one refers to one particular type of polynomial with certain specified boundary conditions.

Interpolation Splines

We specify a spline curve by giving a set of coordinate positions, called control points.

When polynomial sections are fitted so that the curve passes through each control points, the resulting curve is said to interpolate the set of control points.

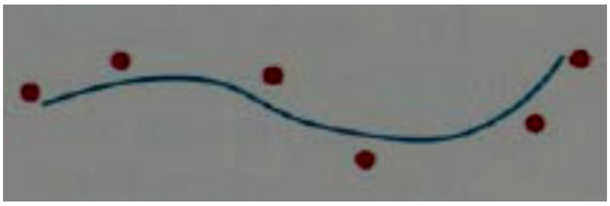


(Figure: 3.14, Interpolation Splines with six control points)

Approximation Splines

When the polynomials are fitted to the general control-point path without necessarily passing through any control point, the result curve is said to approximate the set of control points.

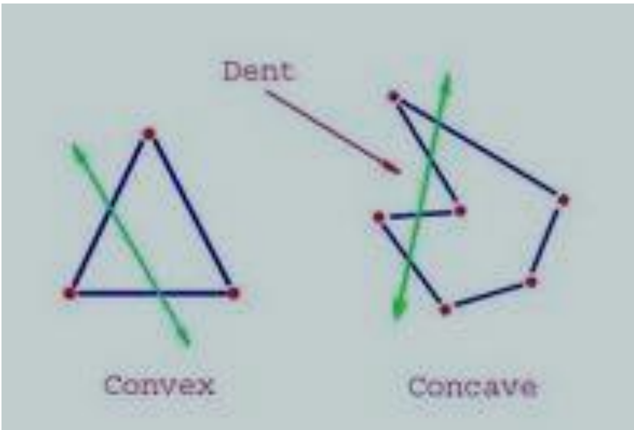
-e.g., Bezier curves, B-spline curves



(Figure: 3.14, Approximation Splines with six control points)

A **convex polygon** is a simple polygon whose interior is a convex set.
The following properties of a simple polygon are all equivalent to convexity:

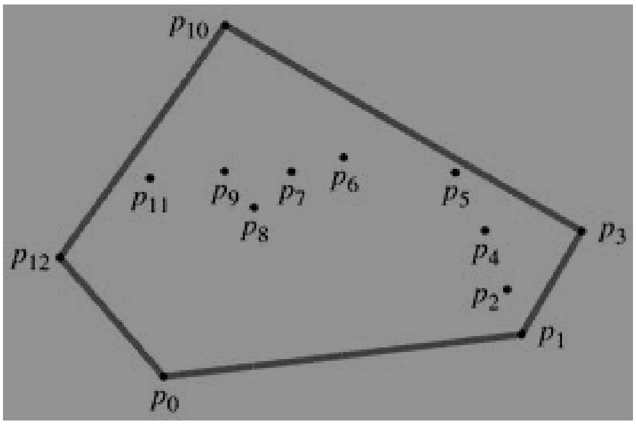
- Every internal angle is less than 180 degrees.
- Every line segment between two vertices remains inside or on the boundary of the polygon.



(Figure: 3.15, Examples for convex and concave polygon)

The convex polygon boundary that encloses a set of control points is called the **convex hull.**

A convex set S is a set of points such that if x, y are in S so is any point on the line between them



(Figure: 3.15, convex hull)

Parametric and Geometric Continuity Conditions

To ensure a smooth transition from one section of a piecewise parametric curve to the next, we impose various continuity conditions at the connection points.

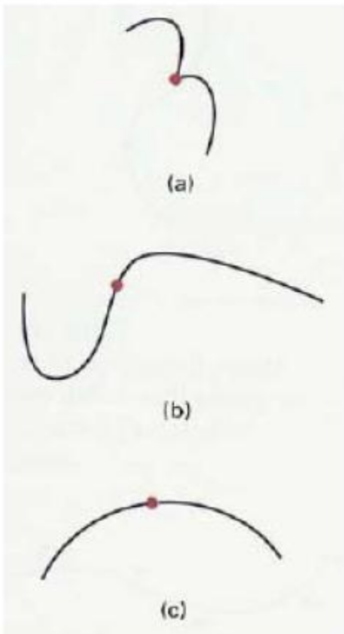
If each section of a spline is described with a set of parametric coordinate functions of the form

x=x(u), y=y(u), z=z(u), u1<=u<=u2

We set parametric continuity by matching the parametric derivatives of adjoining curve sections at their common boundary.

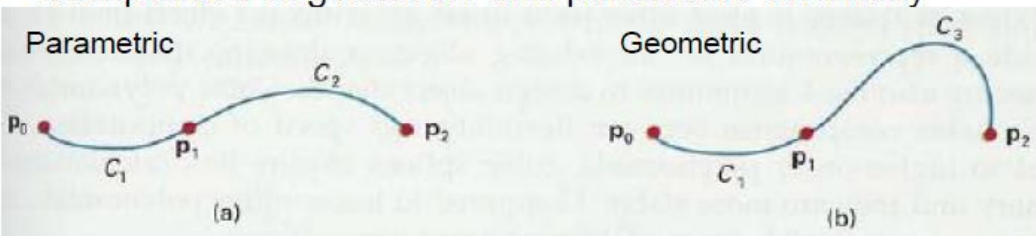
Parametric Continuity Conditions

- Zero-order continuity (C^0):
  - the values of x, y, z evaluated at u_2 for the 1st curve section are equal to the values of x, y, z evaluated at u_1 for the 2nd curve
- First-order continuity (C^1):
  - the 1st derivatives of the coordinate functions for two successive curve sections are equal at their joining point
- Second-order continuity (C^2):
  - both the 1st and 2nd parametric derivatives of the two curve sections are the same at the joining point



Geometric Continuity Conditions

- Zero-order geometric continuity (G^0):
  - the same as zero-order parametric continuity
- First-order geometric continuity (G^1):
  - the parametric 1st derivatives are proportional at the intersection of two successive sections
- Second-order geometric continuity (G^2):
  - both 1st and 2nd parametric derivatives of the two curve sections are proportional at their boundary
- Comparison of geometric and parametric continuity



3.10.1 BEZIER CURVES AND SURFACES

Bezier splines have a number of properties that make them highly useful and convenient for curve and surface design.

- They are also easy to implement.
- **Bezier** splines are widely available in various CAD systems.

The number of control points to be approximated and their relative position determine the degree of the Bezier polynomial.

n + 1 control-point positions: **pk** = (x_k, y_k, z_k), with ***k*** varying from **0** to n. These coordinate points can **be** blended to produce the following position vector ***P(u),*** which describes the path of an approximating Bezier polynomial function between **p0** and pn.

$$\mathbf{P}(u) = \sum_{k=0}^{n} \mathbf{p}_k \, BEZ_{k,n}(u), \qquad 0 \le u \le 1$$

The Bézier blending functions $BEZ_{k,n}(u)$ are the *Bernstein polynomials*:

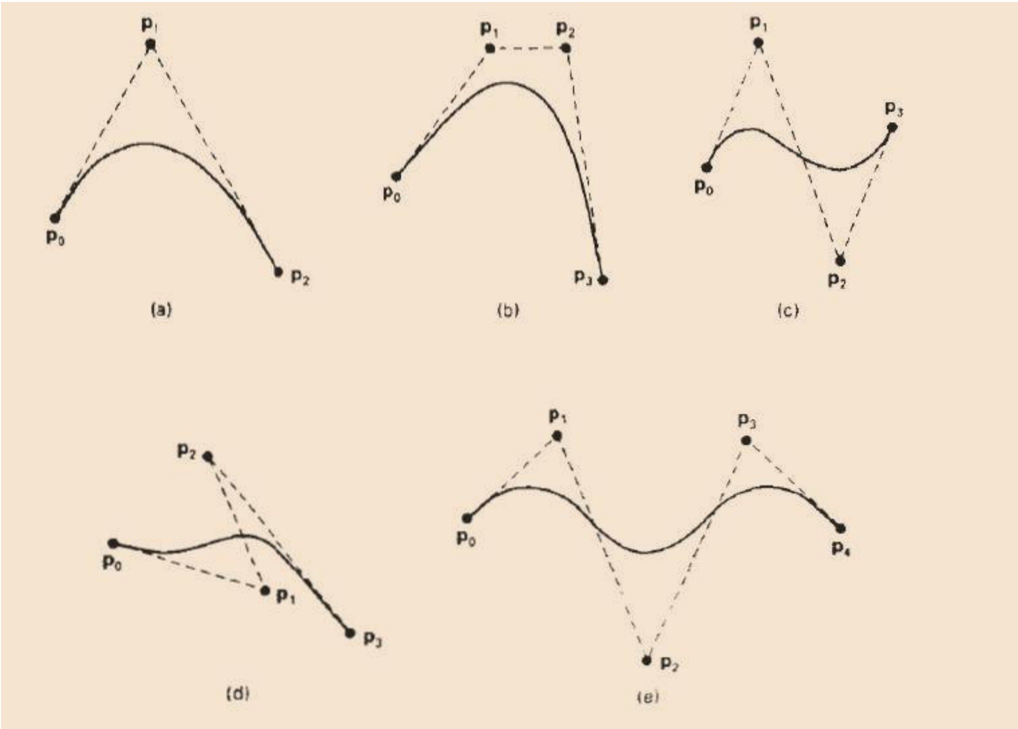
$$BEZ_{k,n}(u) = C(n, k)u^k(1 - u)^{n-k}$$

where the $C(n, k)$ are the binomial coefficients:

$$C(n, k) = \frac{n!}{k!(n-k)!}$$

As a rule, a bezier curve is a polynomial of degree one less than the number of control points used:

Three points generate a parabola; four points a cubic curve, and so forth.



Properties of Bezier Curves

- A very useful property of a Bezier curve is that it always passes through the first and last control points.
- The slope at the beginning of the curve is along the line joining the first two control points, and the slope at the end of the curve is along the line joining the last two endpoints.

- Another important property of any Bezier curve is that it lies within the convex hull (convex polygon boundary) of the control points.

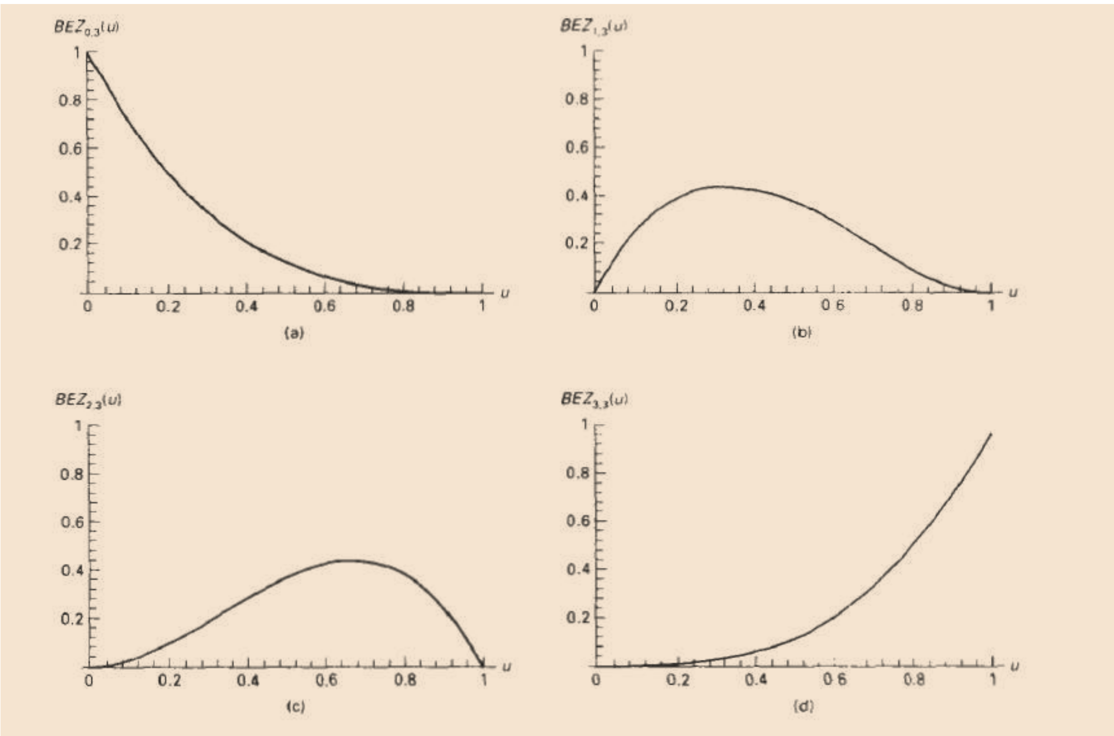
Design Techniques using Bezier curve

- Closed Bezier curves are generated by specifying the first and last control points at the same position.
- Also, specifying multiple control points at a single coordinate position gives more weight to that position.

Cubic Bezier Curves

- Cubic Bezier **curves** are generated with four control Points. The four blending functions for cubic Bezier curves, obtained by substituting ***n*** = 3

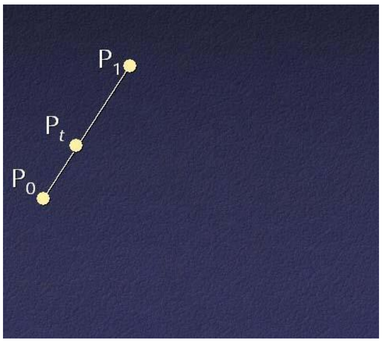
$$BEZ_{0,3}(u) = (1 - u)^3$$
$$BEZ_{1,3}(u) = 3u(1 - u)^2$$
$$BEZ_{2,3}(u) = 3u^2(1 - u)$$
$$BEZ_{3,3}(u) = u^3$$



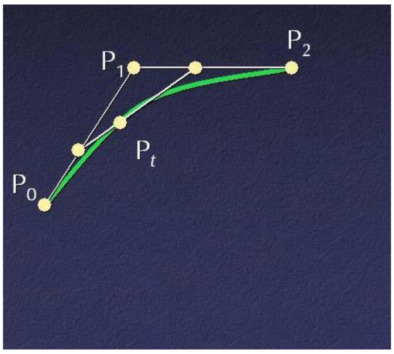
- At ***u*** = 0, the only nonzero blending function is BEZ0, 3, which has the value 1.
- **At u** = 1, the only nonzero function is *BEZ3, 3* with a value of 1 at that point.

Thus, the cubic Bezier curve will always pass through control points **p0** and **p3**

- Specify cubic curve with four control points
- Finding a point on a Bézier Curve.

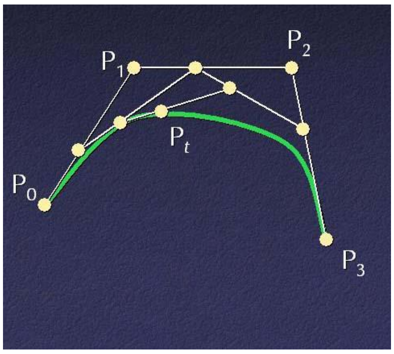


Linear: $P_u = (1-u)P_0 + uP_1$



Quadratic: $Pu = (1-u)[(1-u)P_0 + uP_1] + u[(1-u)P_1 + uP_2]$

$$= (1-u)^2 P_0 + 2u(1-u)P_1 + u^2 P_2$$



Cubic:

$$P_u = (1-u)((1-u)^2 P_0 + 2u(1-u)P_1 + u^2 P_2) + u((1- u)^2 P_1 + 2u(1-u)P_2 + u^2 P_3)$$

$$= (1-u\ )^3 P_0 + 3u(1-u)^2 P_1 + 3u^2 (1-u)P_2 + u^3 P_3$$

BÉZIER SURFACE

- A Bézier surface *S(u,v)* is defined by a grid of control points $P_{i,j}$, where $0 \le i \le m$, $0 \le j \le n$, and $0 \le u \le 1$, $0 \le v \le 1$. The degree is (m,n).

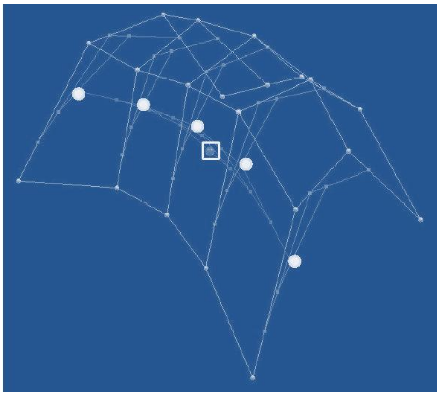
$$S(u,v) = \sum_i \sum_j B_{m,i}(u) B_{n,j}(v) P_{i,j}$$

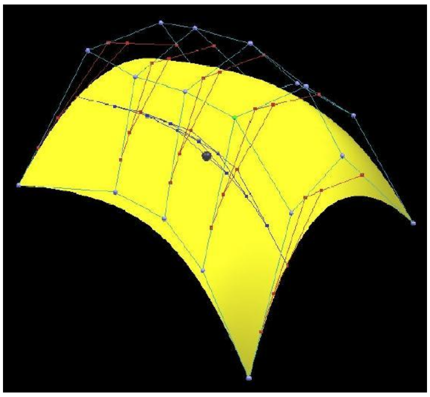
- $B_{m,i}(u)\, B_{n,j}(v)$ are the Bézier basis functions for the surface.

$$B_{m,i}(u) = \frac{m!}{i!(m-i)!} u^i (1-u)^{m-i}$$

$$B_{n,j}(v) = \frac{n!}{j!(n-j)!} v^j (1-v)^{n-j}$$

This Bezier surface has a degree of (4,3).





Important Properties

- *S(u,v)* passes through the control points at the four corners of the control net: $P_{0,0}$, $P_{m,0}$, $P_{m,n}$, $P_{0,n}$.
- Partition of unity: $\sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) = 1$
- Convex hull property: a Bézier surface lies in the convex hull defined by its control net.